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EFFECT OF SECONDARY FLOW ON THE TEMPERATURE FIELD AND PRIMARY FLOW IN A HEATED HORIZONTAL TUBE

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Abstract—Density variations in a fluid flowing in a heated horizontal tube can cause a secondary flow as well as variations in the axial pressure gradient over the tube cross section. The effect of the secondary flow on the temperature field and the primary flow at the outlet of a long electrically heated tube having thick walls of high conductivity is analyzed for the case of large Grashof–Prandtl number for which a thin temperature boundary layer exists near the wall. A model for the flow field is developed which is consistent with the experimental observation that over most of the tube the isotherms are horizontal. By dimensional reasoning it is found that the secondary flow controls the rate of heat transfer. For $\mathbf{P} = 1$ the primary flow also shows a boundary layer behavior while for $\mathbf{P} \rightarrow \infty$ the primary flow is independent of the secondary flow. For constant viscosity and infinite Prandtl number, the Nusselt number is directly proportional to the fourth root of the product of the Grashof and the Prandtl number.

$$\mathbf{N} = C_1 (\mathbf{GP})^{\frac{1}{4}}$$

By integral methods it is estimated that $C_1 = 0.471$. Good agreement is obtained between calculations based on the proposed model and experiment.

NOMENCLATURE

- a, tube radius;
- c, heat capacity of the fluid;
- C_1, $= \mathbf{N}/(\mathbf{GP})^{\frac{1}{4}}$;
- C_2, $= f\mathbf{R}/(\mathbf{GP})^{\frac{1}{4}}$;
- f, Fanning friction factor;
- g, acceleration of gravity;
- $\mathbf{G}$, Grashof number $= \dfrac{a^3 \beta g \rho^2 \Delta T}{\mu^2}$;
- h, heat-transfer coefficient $= q/\Delta T$;
- k, thermal conductivity of the fluid;
- $\mathbf{N}$, Nusselt number $= \dfrac{qa}{\Delta T k}$;
- P, pressure;
- P_0, pressure at the inlet to the heated section;
- P_1, $= P - P_0$;
- $\mathbf{P}$, Prandtl number $= \dfrac{c\mu}{k}$;
- q, rate of heat transfer to the fluid per unit area;
- $\mathbf{R}$, Reynolds number $= \dfrac{a\langle W \rangle \rho}{\mu}$;
- $\mathcal{R}$, $= \mathbf{P}\left(\dfrac{a\mathcal{U}\rho}{\mu}\right)$;
- T, temperature of the fluid;
- T_0, temperature at the inlet;
- T_B, bulk averaged temperature;

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T_w, temperature at the wall;
T_c, temperature in the core;
ΔT, $= T_w - T_B$;
U, velocity in the X-direction;
$\mathscr{U}^+$, characteristic X-velocity in the thermal boundary layer;
u^+, $= U/\mathscr{U}^+ = \left(\frac{aU\rho_0}{\mu}\right)(\mathbf{P})^{\frac{1}{2}}/(\mathbf{G})^{\frac{1}{2}}$;
$\mathscr{U}$, characteristic velocity for the secondary flow in the core of the tube;
u, $= U/\mathscr{U}$;
V, velocity in the Y-direction;
$\mathscr{V}^+$, characteristic velocity for the thermal boundary layer in the Y-direction;
v, $= V/\mathscr{U}$;
v^+, $V/\mathscr{V}^+ = \left(\frac{aV\rho_0}{\mu}\right)(\mathbf{P})^{\frac{3}{4}}/(\mathbf{G})^{\frac{1}{4}}$;
$\langle W \rangle$, bulk averaged velocity;
W, velocity in the Z-direction;
w, $= W/\langle W \rangle$;
X, core coordinate in the vertical direction, boundary layer coordinate in the circumferential direction;
x, dimensionless core coordinate $= X/a$;
x^+, dimensionless boundary coordinate $= X/a$;
Y, core coordinate in the horizontal direction, boundary layer coordinate perpendicular to the wall;
y, dimensionless core coordinate $= Y/a$;
y^+, dimensionless boundary coordinate $= Y(\mathbf{GP})^{\frac{1}{4}}/a$;
Z; coordinate in the axial direction;
z, $= Z/a$.

Greek symbols
β, coefficient of thermal expansion;
δ_T, normalization parameter for the thermal boundary layer $= a(\mathbf{PG})^{-\frac{1}{4}}$;
δ_h, normalization parameter for the hydrodynamic boundary layer;
Δ_T, thickness of the thermal boundary layer;
Δ_T^+, $= \Delta_T/\delta_T$;
θ, angle measured from the bottom of the tube;
μ, viscosity;
ρ, density;
ρ_0, density evaluated at T_0;
ϕ, dimensionless temperature $= \frac{T - T_B}{\Delta T}$;
ϕ_c, $\frac{T_c - T_B}{\Delta T}$;
ψ, stream function.

1. INTRODUCTION

THE CALCULATION of the rate of heat transfer to a fluid flowing laminarly in a heated horizontal tube is complicated because the flow pattern is affected by the variation of fluid properties with temperature. Yang [1] has shown that viscosity variations cause the heat transfer rate to change by as much as fifty per cent from the prediction of Graetz [2] for a undirectional parabolic flow. The recent correlation of heat-transfer measurements by Oliver [3] indicates that the neglect of density variations in the solution of Yang can be serious for fluids of low viscosity for which the heat transfer rate can be larger than the results obtained by Graetz by a factor of three to four.

Density variation in a heated fluid causes a secondary motion that is symmetric about a vertical plane passed through the axis of the tube to be superimposed on the primary flow in the direction of the tube axis. This secondary motion can not only directly increase the rate of heat transfer but also can distort the parabolic velocity profile that would exist for isothermal flow. Additional changes in the axial velocity profiles can occur because changes of the average density in the axial direction are accompanied by a variation of the axial pressure gradient over the cross section of the tube. In fact, if the axial density gradient is large enough the fluid in the top part of the tube will flow backwards for heating [4]. This paper is concerned with obtaining an understanding of the effect of the secondary flow on the temperature field and the primary velocity field.

The system considered is the outlet of a long electrically heated tube having a thick wall of high conductivity. The magnitudes of the secondary velocities increase with an increase in the difference between the wall temperature and the bulk averaged temperatue, to be designated as ΔT. If the heat transfer were carried out using a wall with a constant temperature as might be done by circulating a heat transfer medium in a jacket surrounding the tube, ΔT would change along the length of the tube. A tube heated electrically so that the heat flux per unit length, q, is constant is more suitable to consider the effects of secondary flow since a fully developed condition is approximated downstream for which the bulk temperature of the fluid, T_B, and the wall temperature, T_w, are increasing linearly with distance downstream and for which ΔT is constant. A thick walled tube of high conductivity provides a good conductive path so that the temperature of the inner wall will not vary around the circumference at any axial location.

The fluid enters the heat transfer section with a temperature T_0 and a bulk averaged velocity $\langle W \rangle$. One may consider that either q or ΔT are known at some location where the temperature field is fully developed. The viewpoint in this paper is that ΔT is given. It is desired to calculate the axial velocity profile, the temperature profile, the heat flux and the pressure drop. The bulk averaged temperature T_B can be related to q and $\langle W \rangle$ by using an overall energy balance. Despite the simplifications that result from this particular statement of the problem, the four coupled partial differential equations that define the system are far too complicated to obtain any general solutions. It seems necessary to develop models for the secondary flow field so that the equations can be simplified. This, then, is the principal goal of this paper.

In order to focus upon the effect of the secondary flow the variation of fluid viscosity will be ignored. The dimensionless groups defining the fully developed flow of a fluid of constant viscosity in an electrically heated pipe are the Reynolds number $\mathbf{R}$, the Grashof number, $\mathbf{G}$, the Prandtl number, $\mathbf{P}$, the Nusselt number, $\mathbf{N}$, and the Fanning friction factor, f, where $\mathbf{R}$, $\mathbf{G}$ and $\mathbf{N}$ use the tube radius a and $\mathbf{G}$ uses the temperature difference ΔT. Morton [5] neglected axial density gradients and obtained a perturbation solution for the temperature and velocity field valid for small values of $\mathbf{NG}/\mathbf{R}$. E. del Casal and Gill [6] included effects of both the axial density variation and of the secondary flow in their perturbation solution valid for small values of $\mathbf{NG}/\mathbf{PR}^2$. Both of these schemes are limited to very small values of the heating rate. In order to deal with natural convection effects of the magnitude indicated by the correlation of Oliver [3] it is necessary to seek a solution valid for large values of the product $\mathbf{GP}$, yet not so large that the flow is unstable. Temperature measurements [7], [4] and [8] and dimensional reasoning indicate that a thin temperature boundary layer exists near the wall for large $\mathbf{GP}$. That portion of the fluid external to this boundary layer will be called the core. By considering the core and the boundary layer separately two sets of equations which are much simpler than the original equations are obtained. This boundary layer problem is different from that usually encountered in that the conditions of the external flow, the core, are not given *a priori* but are related to the flow in the boundary layer. The central problems to be faced are to define the conditions in the core and to discover how the core and boundary layer solutions are coupled.

The effect of the temperature field on the flow depends strongly on the value of $\mathbf{P}$; and therefore, two conditions will be considered, $\mathbf{P} = 1$ and $\mathbf{P} \to \infty$. It will be shown for $\mathbf{P} = 1$ that the secondary motion produces a boundary layer behavior in the primary flow similar to that for the temperature field, while for $\mathbf{P} \to \infty$ the secondary flow has no effect on the primary flow.

In the development of the flow model use is made of the experimental observation [7], [4] and [8] that the isotherms in the core are perpendicular to the direction of gravity. As a consequence one expects that for $\mathbf{P} = 1$ the secondary motion has a large upward flow close to the wall and a relatively small downward velocity in the core. For $\mathbf{P} \to \infty$ horizontal isotherms seem consistent either with large or small fluid velocities in the core, and it is necessary to look at the variation of the temperature in the vertical direction to decide on a flow model for the core.

2. EQUATIONS FOR THE FULLY DEVELOPED REGION OF AN ELECTRICALLY HEATED HORIZONTAL TUBE

The equations for the fully developed region of an electrically heated horizontal tube of radius a will now be developed. An X, Y, Z coordinate system will be used with the origin at the center of the pipe, the X-coordinate in the vertical direction, and the Z-coordinate in the direction of flow. Since the flow is fully developed none of the velocity components are changing in the Z-direction. The difference between the bulk averaged temperature at any cross section and the inlet temperature is given as

$$T_B - T_0 = \frac{2qz}{\rho_0 ca\langle W\rangle}. \tag{1}$$

Add T to both sides of (1). Then

$$T - T_0 = \frac{2qz}{\rho_0 Ca\langle W\rangle} + (T - T_B) \tag{2}$$

where $(T - T_B)$ is a function of X and Y and not of Z. The usual approximation that the effect of density variations is manifested in the buoyancy terms and not in the inertia terms is also made. The density is assumed to vary linearly with temperature

$$\frac{\rho}{\rho_0} = 1 - \beta(T - T_0) \tag{3}$$

or after substituting (2)

$$\frac{\rho}{\rho_0} = 1 - \beta\left(\frac{2qz}{\rho_0 Ca\langle W\rangle} + T - T_B\right). \tag{4}$$

The pressure will be taken as the sum of a static pressure at the entrance to the heat transfer section P_0, and a remainder pressure P_1. Since the fluid is assumed to be isothermal at the entrance

$$0 = -\frac{1}{\rho_0}\frac{\partial P_0}{\partial X} - g$$
$$0 = -\frac{1}{\rho_0}\frac{\partial P_0}{\partial Y}. \tag{5}$$

The velocities in the X and Y directions will be made dimensionless with respect to a velocity $\mathscr{U}$, as yet unknown, and the velocity in the Z direction, with respect to $\langle W\rangle$. A dimensionless temperature will be defined as

$$\phi = \frac{T - T_B}{T_w - T_B}.$$

The coordinates are normalized by the radius of the pipe.

The dimensionless form of the energy equation is

$$u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} = -\frac{2w\mathbf{N}}{\mathscr{R}} + \frac{1}{\mathscr{R}}\nabla^2\phi \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad \text{and} \quad \mathscr{R} = \mathbf{P}\left(\frac{a\mathscr{U}\rho_0}{\mu}\right).$$

Since we are looking for a solution for which a temperature boundary layer exists the terms on the left-hand side of (6) representing the convection of heat by the secondary flow must be large compared to the terms representing the conduction of heat everywhere except in a very small region close to the wall. We will therefore be looking for a solution for which $\mathscr{R}$ is a large number. The dimensionless group

$\mathcal{R}$ is a Péclét number defined in terms of the characteristic velocity in the core. It is not independent but is related to the dimensionless groups defined in the Introduction.

The dimensionless form of the Z-momentum equation is

$$\frac{1}{\mathbf{P}}\left(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}\right)=-\frac{a^2}{\mu\langle W\rangle}\frac{1}{\mathcal{R}}\frac{\partial P_1}{\partial z}+\frac{1}{\mathcal{R}}(\nabla^2 w) \quad (7)$$

The behavior of the solution of this equation for large $\mathcal{R}$ depends on the value of $\mathbf{P}$. For $\mathbf{P}=1$ it is seen that the equation for w is similar to that for ϕ and one can expect a solution of the boundary layer type. For $\mathbf{P}/\mathcal{R}\rightarrow\infty$ the convective terms on the left side of the equation will be negligible compared to the viscous terms. One concludes that for $\mathbf{P}/\mathcal{R}\rightarrow\infty$ the secondary flow does not affect the velocity field in the Z-direction.

The secondary flow is described by the equations

$$\frac{\partial u}{\partial \chi}+\frac{\partial v}{\partial y}=0 \quad (8)$$

$$\frac{1}{\mathbf{P}}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{a}{\mathbf{P}\rho_0\mathcal{U}^2}\frac{\partial P_1}{\partial X}+\frac{\mathbf{GP}}{\mathcal{R}^2}\left(\phi+\frac{2zh}{c\langle W\rangle\rho_0}\right)+\frac{1}{\mathcal{R}}(\nabla^2 u) \quad (9)$$

$$\frac{1}{\mathbf{P}}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{a}{\mathbf{P}\rho_0\mathcal{U}^2}\frac{\partial P_1}{\partial Y}+\frac{1}{\mathcal{R}}(\nabla^2 v) \quad (10)$$

where h is the heat-transfer coefficient defined as $q/\nabla T$. Since the equation of conservation of mass (8) is two-dimensional, a stream function can be defined for the secondary flow. The equation for this stream function can be obtained by eliminating the pressure between (9) and (10)

$$\frac{1}{\mathbf{P}}\left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\nabla^2\psi=\frac{\mathbf{GP}}{\mathcal{R}^2}\frac{\partial\phi}{\partial y}+\frac{1}{\mathcal{R}}\nabla^2(\nabla^2\psi) \quad (11)$$

The experimental observation that $\partial\phi/\partial y\rightarrow 0$ over most of the flow field implies that the coefficient $\mathbf{GP}/\mathcal{R}^2$ is of larger order than the coefficients in front of the other two terms. Later it will be shown that for a core with small velocities $\mathcal{R}=(\mathbf{GP})^{\frac{1}{4}}$ and therefore that $\mathbf{GP}/\mathcal{R}^2=(\mathbf{GP})^{\frac{1}{2}}$. For a core with large velocities $\mathcal{R}=(\mathbf{GP})^{\frac{1}{2}}$ and $\mathbf{GP}/\mathcal{R}^2=1$. For $\mathbf{P}=1$ the assumption of large return velocities is inconsistent with the observation that over most of the flow field the isotherms are horizontal. However for $\mathbf{P}\rightarrow\infty$ either estimate for $\mathcal{R}$ implies that $\partial\phi/\partial y\rightarrow 0$.

From equation (9) one concludes that if the isotherms are horizontal over most of the flow field the pressure changes are primarily hydrostatic and not affected by the secondary flow. The viscous terms are of the same magnitude as the buoyancy terms only in very thin regions where the velocity gradients can be quite large. It is of interest to note that the ratio of the inertia to the viscous terms for large $\mathcal{R}$ depends on the magnitude of $\mathbf{P}$. For $\mathbf{P}=1$ the inertia terms will be larger than the viscous terms over most of the flow field except for a viscous boundary layer close to the wall. For $\mathbf{P}/\mathcal{R}\rightarrow\infty$ the convective terms are small compared to the viscous terms over the whole cross section of the pipe and there is no region where viscous terms are of the same magnitude as the inertia terms. This result is in contrast to what is found for a heated vertical plate [9]. The reason for this is that the extent of the flow field for the heated horizontal tube is fixed to that of the tube radius while the heated vertical plate has a field of infinite extent to accommodate the boundary layer. As $\mathbf{P}/\mathcal{R}\rightarrow\infty$ it is necessary that the velocity boundary layer cover a region that is much larger than the tube radius in order that the inertia terms be important.

As can be seen from (11) the term $2zh/(c\langle W\rangle\rho_0)$, that appears in (9) because of density variations in the Z-direction, does not have a direct effect on the secondary flow. It only causes the pressure gradient in the direction of mean flow to vary over the tube cross section. The importance of axial density variations can be estimated by differentiating (9) with respect to Z to obtain

$$\frac{a^2}{\mathbf{P}\rho_0\mathscr{U}^2}\frac{\partial^2 P_1}{\partial Z\partial X} = 2\frac{\mathbf{NG}}{\mathscr{R}^2\mathbf{R}}. \quad (12)$$

If (7) is differentiated with respect to x and (12) is substituted into the resulting equation, it is found that

$$\frac{1}{\mathbf{P}}\frac{\partial}{\partial x}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) = -2\frac{\mathbf{NG}}{\mathbf{PR}^2\mathscr{R}} + \frac{1}{\mathscr{R}}\frac{\partial}{\partial x}(\nabla^2 w). \quad (13)$$

By examining (13) it is seen that the following criteria are established for neglecting density variations

$$\frac{2\mathbf{NG}}{\mathbf{PR}^2} \ll 1 \quad \text{for} \quad \mathbf{P}\to\infty$$
$$\frac{2\mathbf{NG}}{\mathbf{R}^2\mathscr{R}} \ll 1 \quad \text{for} \quad \mathbf{P} = 1. \quad (14)$$

It is to be noted that the axial density variation becomes particularly important for small $\mathbf{R}$; i.e. for small throughputs. From (14) we conclude the effect may be neglected for $\mathbf{P}\to\infty$. It will be shown later that $\mathbf{N} = C_1\mathscr{R}$ for $\mathbf{P} = 1$. We conclude that axial density gradients could be having some effect for $\mathbf{P} = 1$.

By differentiating (10) with respect to Z we find that $\partial P_1/\partial Z$ is not a function of Y. Equation (12) can therefore be integrated to obtain

$$\frac{a}{\rho_0\mathscr{U}^2}\frac{\partial P_1}{\partial Z} = \frac{a}{\rho_0\mathscr{U}^2}\left\langle\frac{\partial P_1}{\partial Z}\right\rangle + \frac{2\mathbf{NGP}}{\mathscr{R}^2\mathbf{R}}x \quad (15)$$

where $\langle\partial P_1/\partial Z\rangle$ is the average value of $\partial P_1/\partial Z$ over the cross section.

3. BOUNDARY LAYER EQUATIONS

In order to treat the boundary layer flow near the wall a coordinate system will be used with the X-axis tangent to the pipe wall the Y-axis perpendicular to the wall and the Z-axis in the direction of mean flow. The usual boundary layer assumption is made that the pressure gradient in the X-direction is independent of Y so that the pressure in the boundary layer equals the pressure in the core evaluated at the wall. Since pressure variations in the core are hydrostatic,

$$\frac{\partial P_1}{\partial X} = \rho_0\beta q(T_c - T_B)\sin\frac{X}{a} + \frac{2Zq\beta g}{ac\langle W\rangle}\sin\frac{X}{a}. \quad (16)$$

The buoyancy and pressure gradient terms in the X-momentum equation are therefore

$$-\frac{\partial P_1}{\partial X} + \rho_0\beta q(T - T_B)\sin\frac{X}{a} + \frac{2Zq\beta g}{ac\langle W\rangle}\sin\frac{X}{a} = \rho_0\beta g(T - T_c)\sin\frac{X}{a}. \quad (17)$$

The boundary layer equations may be written as follows:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad (18)$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} = \beta g(T - T_c)\sin\frac{X}{a} + \frac{\mu}{\rho_0}\frac{\partial^2 U}{\partial Y^2} \quad (19)$$

$$U\frac{\partial W}{\partial X} + V\frac{\partial W}{\partial Y} = -\frac{1}{\rho_0}\frac{\partial P_1}{\partial Z} + \frac{\mu}{\rho_0}\frac{\partial^2 W}{\partial Y^2} \quad (20)$$

$$U\frac{\partial T}{\partial X} + V\frac{\partial T}{\partial Y} + W\frac{2q}{\rho_0 ca\langle W\rangle} = \frac{k}{\rho_0 c}\frac{\partial^2 T}{\partial Y^2} \quad (21)$$

The thermal boundary layer thickness will be assumed of order δ_T. The characteristic length and velocity in the X-direction will be taken as a and $\mathscr{U}^+$. The characteristic velocity in the Y-direction is then obtained from the continuity equation as $(\delta_T/a)\mathscr{U}^+$. The characteristic velocity $\mathscr{U}^+$ and characteristic length

δ_T can be evaluated by assuming that in the thermal boundary layer the viscous and buoyancy terms in the X-momentum equation and convection and conduction terms in the energy equation are of the same magnitude

$$\delta_T = a(\mathbf{PG})^{-\frac{1}{4}}$$

$$\mathscr{U}^+ = \frac{\mu}{\rho_0 a}(\mathbf{G}/\mathbf{P})^{\frac{1}{2}} \tag{22}$$

$$\mathscr{V}^+ = \frac{\mu}{\rho_0 a}(\mathbf{G}/\mathbf{P}^3)^{\frac{1}{4}}$$

The dimensionless forms of the energy balance and the boundary conditions on ϕ are

$$u^+ \frac{\partial \phi}{\partial x^+} + v^+ \frac{\partial \phi}{\partial y^+} + \frac{W\mathbf{N}}{\langle W \rangle (\mathbf{GP})^{\frac{1}{4}}} = \frac{\partial^2 \phi}{\partial y^{+2}} \tag{23}$$

$$\begin{aligned} \phi &= 1 \quad \text{at} \quad y^+ = 0 \\ \phi &= \phi_c \quad \text{at large} \quad y^+ \end{aligned} \tag{24}$$

The heat flux to the wall and therefore the Nusselt number can be evaluated as follows:

$$\mathbf{N} = C_1(\mathbf{PG})^{\frac{1}{4}} \tag{25}$$

$$C_1 = -\frac{1}{\pi}\int_0^{\pi} \frac{\partial \phi}{\partial y}\bigg|_{y^+=0} \mathrm{d}x^+ \tag{26}$$

It will be shown later for $\mathbf{P} = 1$ that $W/\langle W \rangle$ is of order unity and $\mathbf{N}$ is of order $\mathbf{G}^{\frac{1}{4}}$ so that $(W/\langle W \rangle)\mathbf{N}$ is of order $\mathbf{G}^{\frac{1}{4}}$. Since W is not influenced by the secondary flow for $\mathbf{P}/\mathscr{R} \to \infty$ the order of $W/\langle W \rangle$ in the thermal boundary layer is estimated as δ_T/a. The Nusselt number is of order a/δ_T so for $\mathbf{P}/\mathscr{R} \to \infty$, $(W/\langle W \rangle)\mathbf{N}$ is of order unity. Therefore, as $\mathbf{PG} \to \infty$ the term involving the convection of heat in the Z-direction becomes negligibly small in the thermal boundary layer, and it is concluded that the secondary flow controls the rate of heat transfer. The constant C_1 appearing in (25) in general is a function of the parameters of the problem. It will be a constant of order unity if ϕ is only a function of x^+ and y^+. This requires that u^+ and v^+ in the thermal boundary layer and the dimensionless core temperature evaluated at the wall, ϕ_c, be independent of the parameters of the problem. This will be shown to be the case for $\mathbf{P}/\mathscr{R} \to \infty$ if the effect of axial density variation can be neglected.

Before considering the equations for u^+ and v^+ it is necessary to make some general comments about the secondary flow. By using the same type of reasoning as for the heated vertical plate [9] it can be shown that if a region exists in which the inertia and viscous terms are of the same magnitude its extent, δ_h, is given as $(\delta_h/\delta_T) = \mathbf{P}^{\frac{1}{2}}$. For $\mathbf{P} = 1$ the ratio $(\delta_h/\delta_T) = 1$ and as shown in the previous section inertia terms dominate over the viscous terms in the core. One concludes that for $\mathbf{P} = 1$ the boundary layer is a region where viscous, buoyancy and inertia terms are of the same magnitude. In the outer region of the boundary layer, inertia terms dominate since it is necessary to match the boundary layer and core behaviors.

Now let us consider the case of $\mathbf{P} \to \infty$ for a fixed large value of $\mathbf{PG}$, or, according to (22), a fixed small value of (δ_T/a). A region in which the viscous and inertia forces are of the same magnitude would have to be larger than the pipe radius. Clearly this is impossible so the flow at the edge of the thermal boundary layer must be matched to a flow in the core in which viscous forces dominate over inertia forces. From (28) it can be seen for $\mathbf{P}/\mathscr{R} \to \infty$ that $\partial^2 u^+/\partial y^{+2} = 0$ in the matching region. The velocity gradient in the matching region is of order unity when normalized with respect to core parameters $\mathscr{U}$ and a, i.e. $\partial u/\partial y$ is of order unity. The following relation may be written for the matching region

$$\frac{\partial u^+}{\partial y^+} = \frac{\delta_T}{a}\frac{\mathscr{U}}{\mathscr{U}^+}\frac{\partial u}{\partial y}.$$

The ratio $\mathscr{U}/\mathscr{U}^+$ is of order unity or less. Since $\partial u/\partial y$ is of order unity and since $\delta_T/a \to 0$, it follows that $\partial u^+/\partial y^+ \to 0$ in the matching region. Therefore as $\mathbf{P}/\mathscr{R} \to \infty$ the inflection point in the u^+ velocity profile that corresponds to the "edge" of the temperature boundary layer must be approaching either a maximum or a minimum in the secondary flow. If it is approaching a maximum then there is a large return flow in the core. If it is approaching a minimum the velocities in the return flow are much smaller than in the upward flow near the wall. In contrast to the heated vertical plate more than one inflection point can exist in the velocity profile in the heated tube because of the return flow in the core. It will be assumed that the flow is simple enough that no more than two inflections can exist. If there is only one inflection, the matching region will approach the maximum velocity as $\mathbf{P}/\mathscr{R} \to \infty$ as indicated in Fig. 1. The extent and velocity of the upward

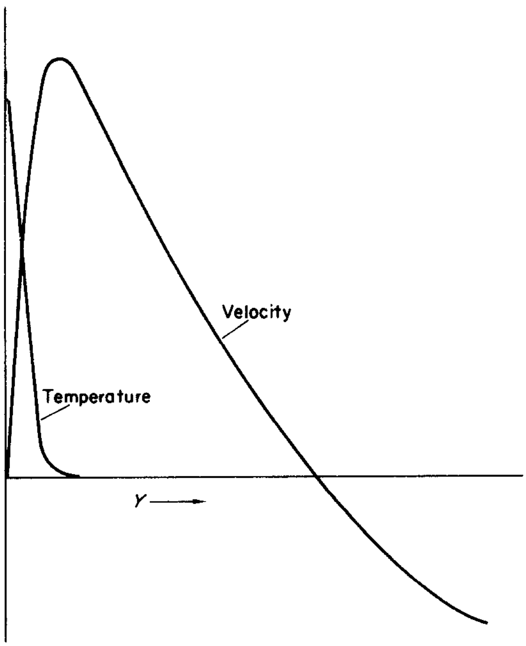

FIG. 1. Velocity and temperature profiles for a heated horizontal tube having a large return flow in the core. Prandtl number is large.

moving flow and return would then be of the same magnitude. Figure 2 shows that the matching region would be near the minimum if two inflections exist in the velocity profile. The inflection point would approach the minimum, and the minimum would approach zero

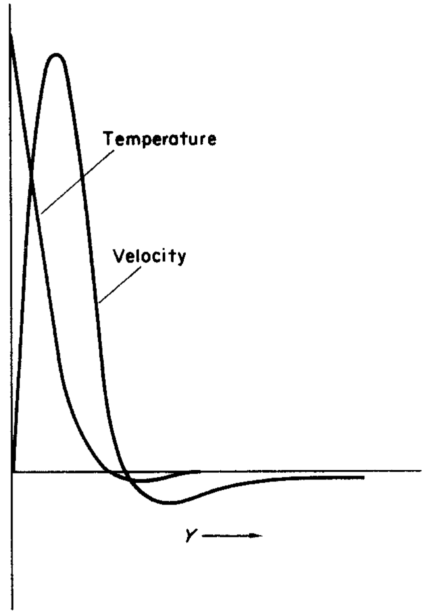

FIG. 2. Velocity and temperature profiles for a heated horizontal tube having a small return flow in the core. Prandtl number is large.

as $\mathbf{P}/\mathscr{R} \to \infty$. The upward moving flow would be confined to the thermal boundary layer and therefore the return flow must occupy a greater portion of the tube and have a much smaller velocity than the upward flow.

If (22) is used to normalize the boundary layer equations describing the secondary flow, (18) and (19), the equations and boundary conditions become

$$\frac{\partial u^+}{\partial x^+} + \frac{\partial v^+}{\partial y^+} = 0 \tag{27}$$

$$\frac{1}{\mathbf{P}}\left(u^+ \frac{\partial u^+}{\partial x^+} + v^+ \frac{\partial u^+}{\partial y^+}\right) = (\phi - \phi_c)\sin x^+ + \frac{\partial^2 u^+}{\partial y^{+2}} \tag{28}$$

$$\begin{aligned} u^+ = v^+ = 0, \phi = 1 \quad &\text{at} \quad y^+ = 0 \\ \phi = \phi_c \quad &\text{at large} \quad y^+ \end{aligned} \tag{29}$$

If the upflow is confined to the thermal boundary layer, it is appropriate to assume $u^+ = 0$ at large

y^+ since for large **GP** the return velocities will be negligible compared to the boundary layer velocities. For a profile at $\mathbf{P}/\mathscr{R} \to \infty$, which would have a large return flow, it is appropriate to apply (28) only within the thermal boundary layer since the rest of the flow would not be of the boundary layer type. The boundary condition would then be u^+ is a maximum at large y^+. For $\mathbf{P}/\mathscr{R} \to \infty$ the inertia terms in (28) are negligible and the differential equations and boundary conditions describing the variation of u^+ and v^+ within the thermal boundary layer contain no parameters of the problem provided ϕ_c is a function only of x^+ and y^+.

To complete the discussion of the boundary layer equations the Z-momentum balance will now be treated. As was shown in the previous section viscous terms dominate for $\mathbf{P}/\mathscr{R} \to \infty$ and therefore, a boundary layer analysis is not appropriate. For $\mathbf{P} = 1$ inertia terms dominate in the core and a boundary layer exists near the wall where the inertia and viscous terms are of the same magnitude. The velocity with which to normalize the W-term in the boundary layer is $\langle W \rangle$. The dimensionless form of (20) is then given as

$$\frac{1}{\mathbf{P}}\left(u^+\frac{\partial w^+}{\partial x^+} + v^+\frac{\partial w^+}{\partial y^+}\right) = -\frac{a^2}{\mu(\mathbf{GP})^{\frac{1}{2}}}\frac{\partial P_1}{\partial Z} + \frac{\partial^2 w^+}{\partial y^{+2}} \quad (30)$$

$$\begin{aligned} w^+ &= 0 \quad &\text{at } y^+ = 0 \\ w^+ &= w_c \quad &\text{at large } y^+ \end{aligned} \quad (31)$$

The term w_c is the solution of the equations describing the core flow evaluated at the wall. A force balance that equates the average axial pressure drop to the resisting force at the wall gives

$$-\left\langle\frac{\partial P_1}{\partial Z}\right\rangle = \frac{2\mu\langle W\rangle}{\pi a^2}(\mathbf{PG})^{\frac{1}{4}}\int_0^\pi \left.\frac{\partial w^+}{\partial y^+}\right|_{y^+=0} \mathrm{d}x^+. \quad (32)$$

The Fanning friction factor is defined as

$$f = -\frac{a\langle \partial P_1/\partial Z\rangle}{\rho\langle W\rangle^2} \quad (33)$$

so from (32)

$$f = C_2\frac{(\mathbf{PG})^{\frac{1}{4}}}{\mathbf{R}} \quad (34)$$

where

$$C_2 = \frac{2}{\pi}\int_0^\pi \left.\frac{\partial w^+}{\partial y^+}\right|_{y^+=0} \mathrm{d}x^+. \quad (35)$$

If (32) is substituted into (30) it is seen that the first term on the right side of (30) is negligible if the variation of $\partial P_1/\partial Z$ over the cross section due to axial density gradients is not much larger than $\langle \partial P_1/\partial Z\rangle$. Then the equation describing the variation of w^+ in the boundary layer is the same as the energy equation (23).

4. CORE EQUATIONS

Now that the parameterization of the boundary layer equations has been completed it is possible to return to the discussion of the equations describing flow in the core. If the upflow is confined to the thermal boundary layer the characteristic velocity of the fluid in the core can be calculated by equating the mass flow in the thermal boundary layer, $\delta_T\mathscr{U}^+$, to the mass flow down in the core, $a\mathscr{U}\sin x^+$.

$$\begin{aligned} \mathscr{U} &= \frac{\mu}{a\rho_0}(\mathbf{G}/\mathbf{P}^3)^{\frac{1}{4}} \\ \mathscr{R} &= (\mathbf{GP})^{\frac{1}{4}}. \end{aligned} \quad (36)$$

If the extent of the regions of upflow and downflow are the same, the characteristic velocity for the core is the same as that for the thermal boundary layer.

$$\mathscr{U} = \frac{\mu}{a\rho_0}(\mathbf{G}/\mathbf{P})^{\frac{1}{2}} \quad (37)$$

$$\mathscr{R} = (\mathbf{GP})^{\frac{1}{2}}. \quad (38)$$

The secondary flow in the core is described by (11). The first term on the right side of this equation is the production of vorticity while the second is the diffusion of vorticity. For $\mathbf{P} = 1$ and large $\mathbf{GP}$ the diffusion terms are small in comparison to the convection terms while for $\mathbf{P}/\mathcal{R} \to \infty$ the diffusion terms dominate the convection terms. Even though $\partial\phi/\partial y \to 0$, the production of vorticity cannot be neglected since the coefficient in front of $\partial\phi/\partial y$ is large. The solution of (11) seems to require a knowledge of the variation of ϕ in the core to a high order of accuracy. It is not evident how to proceed with this solution. Fortunately the calculation of the boundary layer properties, the heat transfer rate and the first order variation of ϕ in the core does not depend on a detailed knowledge of the stream function in the core.

Since $\mathcal{R}$ is a large number, the equation for the variation of ϕ in the core, (6), becomes

$$u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} = -\frac{2w\mathbf{N}}{\mathcal{R}}. \quad (39)$$

Making use of (25) and the observation that $\partial\phi/\partial y \to 0$ in the core

$$u\frac{\partial\phi}{\partial x} = -\frac{2wC_1(\mathbf{GP})^{\frac{1}{4}}}{\mathcal{R}}. \quad (40)$$

For the case of large return flow $\mathcal{U}$ is given by (37). Then (40) indicates $\partial\phi/\partial x \to 0$ and it is expected that the core is isothermal. That is, $\phi_c = 0$. If the return flow in the core is small, $\mathcal{U}$ is given by (36) and

$$u\frac{\partial\phi}{\partial x} = -2wC_1. \quad (41)$$

Since C_1 is a constant of order unity it is concluded from (41) that the variation of the core temperature in the vertical direction is of order unity if the upward flow is confined to a small region close to the wall. Since $\partial\phi/\partial x$ is independent of y, (41) can be integrated over the cross section at a fixed value of x.

$$\frac{\partial\phi}{\partial x}\int_0^{\sqrt{(1-x^2)}} u\,dy = -C_1\int_0^{\sqrt{(1-x^2)}} w\,dy. \quad (42)$$

A mass balance at any height in the pipe reveals that

$$-\int_0^{\sqrt{(1-x^2)}} u\,dy = \int_0^{\infty} u^+\,dy^+ = B^+. \quad (43)$$

After substituting in (42),

$$B^+\frac{\partial\phi}{\partial x} = C_1\int_0^{\sqrt{(1-x^2)}} w\,dy. \quad (44)$$

The above differential equation is dependent on the secondary flow in the core only insofar as it is affecting the integral involving w. Since the mixed average value of ϕ in the core must equal zero, an integral condition on ϕ can be given which has the same role as specifying a boundary condition for (44).

$$\int_{-1}^{1}\int_0^{\sqrt{(1-x^2)}} w\phi\,dy\,dx = 0. \quad (45)$$

For $\mathbf{P}/\mathcal{R} \to \infty$ equation (7) can be integrated to give

$$w = 2(1 - x^2 - y^2) + \frac{1}{4}\frac{\mathbf{NG}}{\mathbf{PR}^2}(1 - x^2 - y^2)\cos x\sqrt{(a^2 + y^2)}. \quad (46)$$

If the effect of axial density gradients can be ignored, the second term on the right side of (41) is zero, and the variation of ϕ in the core is independent of the secondary flow and of the parameters of the problem.

For $\mathbf{P} \ll \mathbf{GP}$ the viscous terms in (7) may be neglected and if (15), (32) and (35) are substituted

$$\frac{1}{\mathbf{P}}\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) = \frac{C_2(\mathbf{PG})^{\frac{1}{4}}}{\mathcal{R}} - \frac{2\mathbf{NG}}{\mathcal{R}\mathbf{R}^2\mathbf{P}}x. \quad (47)$$

For $\mathbf{P} = 1$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = C_2 - \frac{2C_1\mathbf{G}}{\mathbf{R}^2}x. \quad (48)$$

Since the variation of u and v in the core is not known, it seems best at present to rely on experiment to determine an approximate solution to (48). The experiments of Y. Mori and his co-workers [7] suggest that for $\mathbf{P} = 1$ $\partial w/\partial y \to 0$ in the core. This is not surprising since (48) and (39) are quite similar. It follows that w may be approximated by

$$B^+\frac{\partial w}{\partial x} = -\sqrt{(1-x^2)}\left(C_2 - \frac{2C_1\mathbf{G}}{\mathbf{R}^2}x\right). \quad (49)$$

Since the average value of w equals unity the following integral condition can be established:

$$\int_{-1}^{1}\int_{0}^{\sqrt{(1-x^2)}} w\,\mathrm{d}y\,\mathrm{d}x = 1. \quad (50)$$

5. STATEMENT OF BOUNDARY LAYER PROBLEM, $\mathbf{P}/\mathcal{R} \to \infty$

The equations developed in the previous sections relating to the solutions of the boundary layer problem will now be summarized for $\mathbf{P}/\mathcal{R} \to \infty$ and for the case of negligible effect of axial density gradients. It should be recalled that x and y are core coordinates that are aligned with the vertical and horizontal and which are made dimensionless with respect to the pipe radius. The term x^+ is the boundary layer coordinate tangent to the wall. It is dimensionless with respect to a. The perpendicular distance from the wall made dimensionless with respect to $\delta_T = a(\mathbf{PG})^{-\frac{1}{4}}$ is designated by y^+. The coordinates x^+ and x are related as follows:

$$x = -\cos x^+. \quad (51)$$

No boundary layer solution is needed to define the variation of the Z-component of velocity for $\mathbf{P}/\mathcal{R} \to \infty$ since it is unaffected by the secondary flow.

$$w = 2(1 - x^2 - y^2). \quad (52)$$

The pressure drop is given by Poiseuille's law.

When the velocity boundary layer is confined to the thermal boundary layer as depicted in Fig. 2, the boundary layer equations are as follows:

$$u^+\frac{\partial\phi}{\partial x^+} + v^+\frac{\partial\phi}{\partial y^+} = \frac{\partial^2\phi}{\partial y^{+2}} \quad (53)$$

$$\frac{\partial u^+}{\partial x^+} + \frac{\partial v^+}{\partial y^+} = 0 \quad (27)$$

$$\frac{\partial^2 u^+}{\partial y^{+2}} + (\phi - \phi_c)\sin x^+ = 0 \quad (54)$$

$$u^+ = v^+ = 0, \quad \phi = 1 \quad \text{at} \quad y^+ = 0$$

$$u^+ = 0, \quad \phi = \phi_c \quad \text{at large} \quad y^+. \quad (55)$$

The core temperature evaluated at the wall, ϕ_c, is given by the following equations:

$$B^+\frac{\partial\phi_c}{\partial x} = C_1\int_0^{\sqrt{(1-x^2)}} w\,\mathrm{d}y \quad (44)$$

$$B^+ = \int_0^\infty u^+\,\mathrm{d}y^+ \quad (43)$$

$$\int_{-1}^{1}\int_{0}^{\sqrt{(1-x^2)}} w\phi_c\,\mathrm{d}y\,\mathrm{d}x = 0. \quad (45)$$

The Nusselt number can be calculated from the solution of the above equations for the thermal boundary layer.

$$\mathbf{N} = C_1(\mathbf{PG})^{\frac{1}{4}} \quad (25)$$

$$C_1 = -\frac{1}{\pi}\int_0^\pi \frac{\partial\phi}{\partial y^+}\bigg|_{y^+=0}\mathrm{d}x^+ \quad (26)$$

Since no parameters of the problem appear in the differential equations and boundary conditions, it may be concluded that C_1 is a number of order unity. The value of C_1 is determined by solving the above equations. This solution requires an iteration procedure. One possibility is to assume the function $\phi_c(x)$ and then solve (53), (27), (54) and (55). From this solution a new $\phi_c(x)$ can be calculated from (52), (44), (43) and (45) and the procedure is then repeated.

Another method of iterating is initially to assume a value of C_1. If the core temperature at $x = 0$ is used as the reference temperature, the boundary condition for (44) is $\phi_c = 0$ at $x = 0$. The differential equations for the temperature variation in the core and the boundary layer can then be solved simultaneously. After the solution has been completed, the reference temperature is calculated from (45) and a new value of C_1 from (26). The calculation is then repeated.

If the regions of upward and downward flow are of the same magnitude as shown in Fig. 1, the core temperature ϕ_c, is equal to zero. The defining differential equations for the thermal boundary layer are as follows

$$u^+ \frac{\partial \phi}{\partial x^+} + v^+ \frac{\partial \phi}{\partial y^+} = \frac{\partial^2 \phi}{\partial y^{+2}} \tag{56}$$

$$\frac{\partial u^+}{\partial x^+} + \frac{\partial v^+}{\partial y^+} = 0 \tag{27}$$

$$\frac{\partial^2 u^+}{\partial y^{+2}} + \phi \sin x^+ = 0 \tag{57}$$

$$u^+ = v^+ = 0, \quad \phi = 1 \quad \text{at} \quad y^+ = 0 \tag{58}$$

$$\frac{\partial u^+}{\partial y^+} = 0, \quad \phi = 0 \quad \text{at large} \quad y^+.$$

Again, since no parameters appear in the above set of differential equations, C_1 is a constant of order unity. The value of C_1 will be the same as that calculated for a heated cylinder in a fluid of infinite extent.

6. STATEMENT OF BOUNDARY LAYER PROBLEM, P = 1

The boundary layer equations for $\mathbf{P} = 1$ are as follows:

$$u^+ \frac{\partial \phi}{\partial x^+} + v^+ \frac{\partial \phi}{\partial y^+} = \frac{\partial^2 \phi}{\partial y^{+2}} \tag{56}$$

$$u^+ \frac{\partial w}{\partial x^+} + v^+ \frac{\partial w}{\partial y^+} = \frac{\partial^2 w}{\partial y^{+2}} \tag{59}$$

$$\frac{\partial u^+}{\partial x^+} + \frac{\partial v^+}{\partial y^+} = 0 \tag{27}$$

$$u^+ \frac{\partial u^+}{\partial x^+} + v^+ \frac{\partial u^+}{\partial y^+} = (\phi - \phi_c) \sin x^+ + \frac{\partial^2 u^+}{\partial y^{+2}} \tag{60}$$

$$u^+ = v^+ = w = 0, \quad \phi = 1 \quad \text{at} \quad y^+ = 0$$

$$u^+ = 0, \quad w = w_c, \quad \phi = \phi_c \quad \text{at large} \quad y^+. \tag{61}$$

The equations describing $w_c(x)$ and $\phi_c(x)$ are as as follows:

$$B^+ \frac{\partial w_c}{\partial x} = -\sqrt{(1 - x^2)} \;\; C_2 - \frac{2C_1 \mathbf{G}}{\mathbf{R}^2} x \tag{49}$$

$$B^+ \frac{\partial \phi_c}{\partial x} = C_1 w_c \sqrt{(1 - x^2)} \tag{62}$$

$$\int_{-1}^{1} w_c \phi_c \sqrt{(1 - x^2)}\, dx = 0 \tag{45}$$

$$\int_{-1}^{1} w_c \sqrt{(1 - x^2)}\, dx = 1. \tag{50}$$

The above equations are based on an approximation, $\partial w_c/\partial y \cong 0$, which is justified only by experimental observation. The equations for the Nusselt number and friction factor are

$$f = C_2 \frac{\mathbf{G}^{\frac{1}{4}}}{\mathbf{R}} \tag{34}$$

$$C_2 = \frac{2}{\pi} \int_0^{\pi} \left. \frac{\partial w}{\partial y^+} \right|_{y^+ = 0} dx^+ \tag{35}$$

$$\mathbf{N} = C_1 \mathbf{G}^{\frac{1}{4}} \tag{25}$$

$$C_1 = -\frac{1}{\pi} \int_0^{\pi} \left. \frac{\partial \phi}{\partial y^+} \right|_{y^+ = 0} dx^+. \tag{26}$$

The coefficients C_1 and C_2 can be calculated by solving the thermal boundary layer and core equations for ϕ and w. The solution of these equations requires an iterative procedure whereby one initially either assumes $w_c(x)$ and $\phi_c(x)$

or assumes C_1 and C_2. If axial density gradients can be neglected, no parameters appear in the differential equations and boundary conditions defining w and ϕ, and C_1 and C_2 are constants of order unity.

7. APPROXIMATE SOLUTION FOR $P/\mathscr{R} \to \infty$ AND FOR NEGLIGIBLE EFFECT OF AXIAL DENSITY GRADIENTS

A solution will now be presented for $\mathbf{P}/\mathscr{R} \to \infty$ and for negligible effect of axial density gradients. Experiments by Siegwarth [8] with ethylene glycol showed a large variation of ϕ in the core in the vertical direction. These results indicate that the equations developed for the case of small core velocities are the appropriate ones. These equations will be solved approximately in that only the integral form of (53) will be satisfied,

$$\frac{d}{dx^+}\int_0^{\Delta^+} u^+\phi\, dy^+ - \phi_c\int_0^{\Delta^+}\frac{\partial u^+}{\partial x^+}dy^+ = \frac{\partial\phi}{\partial y^+}\bigg|_0^{\Delta^+}. \tag{63}$$

On the basis of the arguments presented following equation (26), the thickness of velocity boundary layer will be taken equal to the thickness of the thermal boundary layer. The following expressions for the velocity and temperature,

$$u^+ = -(1-\phi_c)\Delta^{+2}\sin\ x^+ \left[-\frac{3}{20}\frac{y^+}{\Delta^+} + \frac{1}{2}\left(\frac{y^+}{\Delta^+}\right)^2 - \frac{1}{2}\left(\frac{y^+}{\Delta^+}\right)^3 + \frac{1}{4}\left(\frac{y^+}{\Delta^+}\right)^5 - \frac{1}{10}\left(\frac{y^+}{\Delta^+}\right)^6\right], \tag{64}$$

$$\phi = 1 - 3(1-\phi_c)\frac{y^+}{\Delta^+} + 5(1-\phi_c)\left(\frac{y^+}{\Delta^+}\right)^3 - 3(1-\phi_c)\left(\frac{y^+}{\Delta^+}\right)^4 \tag{65}$$

satisfy (57) and the boundary conditions,

$$u^+ = 0, \quad \phi = 1, \quad \frac{\partial^2\phi}{\partial y^{+2}} = 0 \quad \text{at} \quad y^+ = 0$$
$$u^+ = \frac{\partial u^+}{\partial y^+} = \frac{\partial\phi}{\partial y^+} = 0, \quad \phi = \phi_c \quad \text{at} \quad y^+ = \Delta^+. \tag{66}$$

They will satisfy (63) if Δ^+ is defined by the following differential equation:

$$\frac{d\Delta^+}{dx^+} = \left[-0.00256(1-\phi_c)\Delta^{+2}\sin x^+\frac{d\phi_c}{dx^+} - 0.001694(1-\phi_c)^2\Delta^{+3}\cos x^+ + \frac{3(1-\phi_c)}{\Delta^+}\right] \Big/ 0.00508(1-\phi_c)^2\Delta^{+2}\sin x^+. \tag{67}$$

The boundary condition for the solution of (67) is obtained by assuming symmetry. Equation (67) is solved at $x^+ = 0$ using $d\Delta^+/dx^+ = d\phi_c/dx^+ = 0$ to obtain

$$(1-\phi_c)^{\frac{1}{4}}\,\Delta^+ = 6.49 \quad \text{at} \quad x^+ = 0. \tag{68}$$

From the definition of ϕ and y^+ it follows from (25) that the local Nusselt number is $[3(1-\phi_c)/\Delta^+](\mathbf{GP})^{\frac{1}{4}}$. The constant C_1 appearing in (25) is therefore given as

$$C_1 = \frac{2}{\pi}\int_0^{\pi}\frac{3(1-\phi_c)}{\Delta^+}dx^+. \tag{69}$$

The dimensionless volumetric flow in the boundary layer is calculated from (64) using the equation

$$B^+ = \int_0^{\Delta^+} u^+\, dy^+. \tag{70}$$

Equations (52) and (70) are substituted into (44) and the resulting differential equation is written in terms of boundary layer coordinates by using (51) to obtain the equation defining how ϕ_c varies with x^+.

$$0.00595(1-\phi_c)\Delta^{+3}\frac{d\phi_c}{dx^+} = \frac{4}{3}C_1(1-\cos^2 x^+)^{\frac{1}{4}}. \tag{71}$$

The integral condition (45) is used instead of a boundary condition for (71).

Equations (67), (68), (69), (71) and (45) define C_1 and the variation of Δ^+ and ϕ_c with x^+. An iteration procedure has been used to solve these equations on the computer [11]. It is found that

$$\mathbf{N} = 0{\cdot}471(\mathbf{GP})^{\frac{1}{4}}. \tag{72}$$

The computed variation of B^+, Δ^+ and ϕ_c with θ is shown in Table 1. It is to be noted that the dimensionless boundary layer thickness is relatively constant from $\theta = 0°$ to $\theta = 145°$. After 145° it increases rather rapidly. The shape of the velocity and temperature profiles predicted by the assumed relations (64) and (65) are shown in Fig. 3. It is to be noted that the temperature profile has a minimum at a value of y^+/δ^+ less than unity. The measurements and computer solutions of Siegwarth [8] indicate a similar behavior over a large portion of the tube circumference.

Table 1. *Results of calculations for* $\mathbf{P} \rightarrow \infty$

x^+	θ	x	Δ^+	ϕ_c	B^+	
0	0	−1·000	5·77	−0·595	0	
0·200	11·46°	−0·980	5·77	−0·595	0·362	
0·400	22·92°	−0·921	5·81	−0·591	0·723	
0·600	34·38°	−0·825	5·85	−0·576	1·062	
0·800	45·85°	−0·697	5·90	−0·540	1·350	
1·000	57·31°	−0·540	5·94	−0·476	1·498	
1·200	68·77°	−0·362	5·97	−0·376	1·621	← Max B^+
1·400	80·23°	−0·170	5·98	−0·241	1·560	
1·600	91·69°	0·029	5·99	−0·073	1·375	
1·800	103·15°	0·227	6·01	0·121	1·104	
2·000	114·61°	0·416	6·06	0·332	0·802	
2·200	126·07°	0·588	6·20	0·543	0·524	
2·400	137·54°	0·737	6·61	0·733	0·310	
2·600	149·00°	0·857	7·73	0·863	0·187	
2·800	160·40°	0·942	10·79	0·918	0·204	
3·000	171·92	0·990	17·19	0·923	0·328	
3·130	179·37	1·000	41·84	0·923	0·388	
3·140	179·94	1·000	85·19	0·923	0·450	

These calculations are in good agreement with results of experiments performed with ethylene glycol [8, 11]. Calculated horizontal and vertical core temperature profiles are compared in Figs. 4 and 5 with the measurements made by Siegwarth [8] at $(\mathbf{GP})^{\frac{1}{4}} = 51{\cdot}3$ and $\mathbf{P} = 70{\cdot}7$. It is to be noted that the measured wall temperatures at the top and the bottom of the tube are not the same since the tube wall did not supply a good enough conductive path to equalize the temperature completely.

In Fig. 6 the calculated variation of the local Nusselt number around the circumference of the tube is compared with measurements of the temperature gradient at the wall obtained by Siegwarth [8] at $(\mathbf{GP})^{\frac{1}{4}} = 51{\cdot}3$ and $\mathbf{P} = 70{\cdot}7$. The agreement is within the accuracy of the measurements. The local Nusselt number decreases because of an increase in the thickness of the temperature boundary layer or because of a decrease in the temperature drop through the boundary layer brought about by changes in the core temperature. The change of the local heat-transfer rate can therefore be explained in terms of the calculated results in Table 1. The local heat transfer rate does not vary much from $\theta = 0°$ to $\theta = 60°$ because the boundary layer thickness and ϕ_c are relatively constant over this region. The decrease in the local heat transfer rate from $\theta = 60°$ to $\theta = 145°$

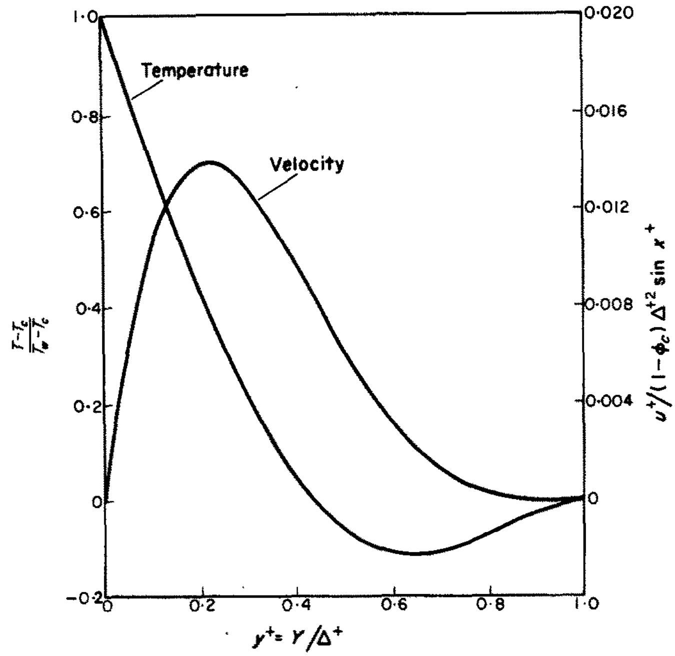

Fig. 3. Temperature and velocity profile assumed for the boundary layer.

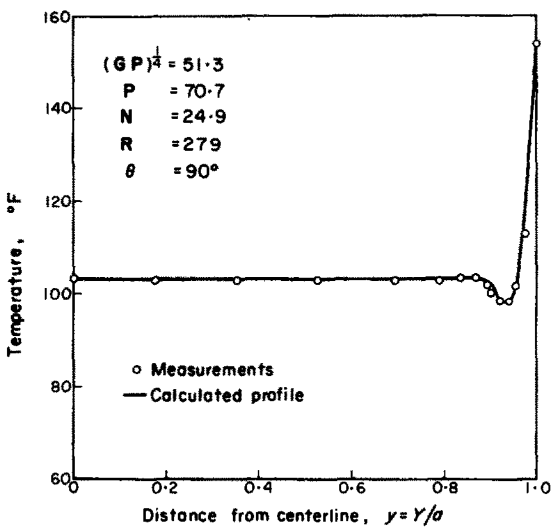

Fig. 4. Comparison of calculated and measured horizontal temperature profiles.

appears to be mainly due to the large increase in ϕ_c. The decrease from $\theta = 145°$ to $\theta = 180°$ is due to changes both in the boundary layer thickness and in ϕ_c.

Measured values of the average heat transfer coefficient around the wall of the pipe made by Readal [11] for $NG/PR^2 = 0·14$–$0·67$ are compared with equation (72) in Fig. 7. The physical properties are evaluated at the bulk averaged temperature of the fluid. This would be close to the average temperature in the boundary layer in the region from $\theta = \theta°$ to $\theta = 60°$ where a large portion of the heat transfer occurs. If the wall temperature had been used, the measurements fall as much as 15 per cent below the calculated line.

8. CONCLUDING REMARKS

The entire analysis presented in this paper is based on the assumption of the existence of a temperature boundary layer. Although a large

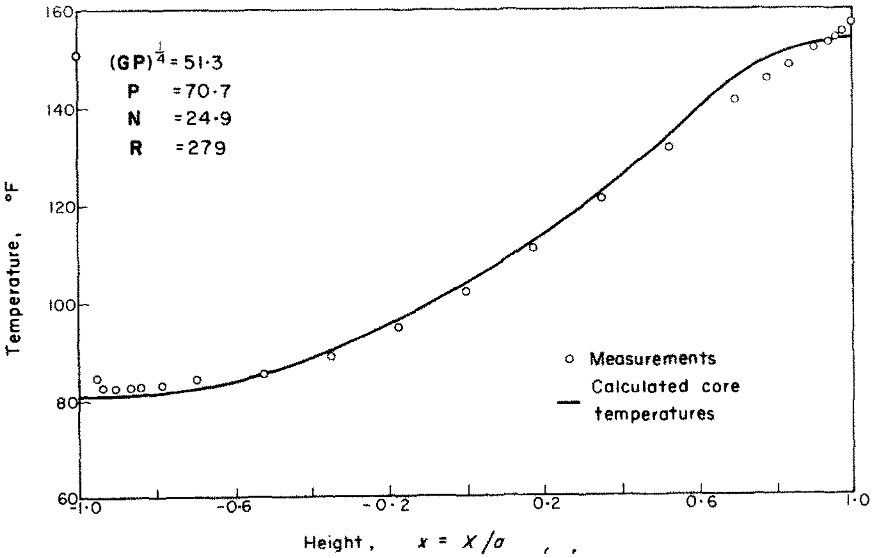

FIG. 5. Comparison of calculated and measured core temperatures.

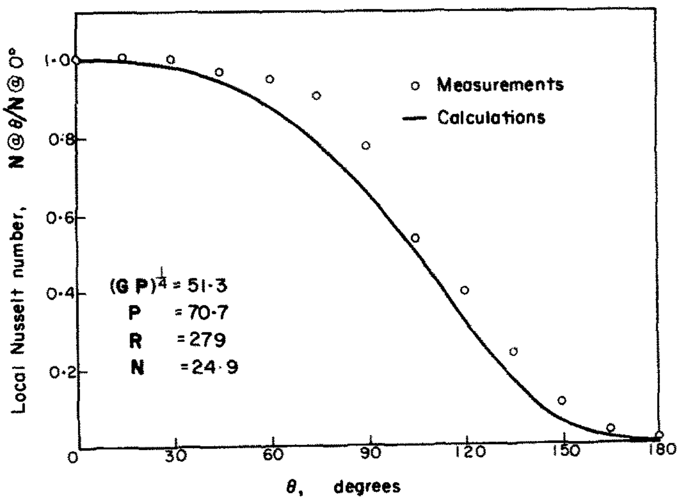

FIG. 6. Comparison of calculates and measured local Nusselt numbers.

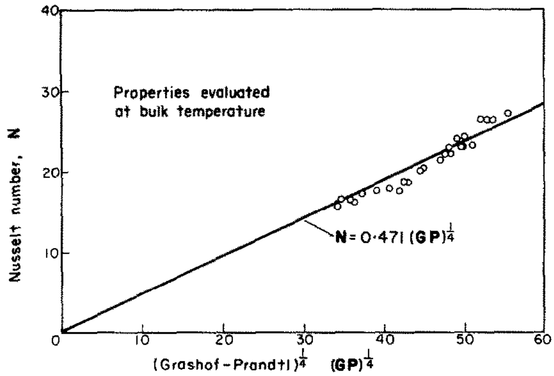

FIG. 7. Comparison of calculated and measured Nusselt numbers.

value of **GP** insures its existence over most of the circumference, it is possible that the boundary layer assumption breaks down at the top of the tube where $T_w - T_c$ can be quite small. In this region the conduction of heat could be important in the core as well as close to the wall. However, it is not likely that this will cause any serious error in the calculation of the heat-transfer coefficient since as shown in Fig. 6, the heat transferred to the fluid is relatively small at the top of the tube.

A case where the non uniformity of the boundary layer solution could be more significant is that of a wall boundary condition of constant heat flux rather than constant temperature. If convection effects become small in the top of the tube, then a very large temperature gradient will exist in the top portion of the core in order to accommodate the heat flux. This large change in T_c will be accompanied by an equally large variation in T_w, so $T_c - T_w$ is small.

The equations presented in this paper have been developed by using the experimental observation that the temperature isotherms in the core are horizontal. This implied that for $\mathbf{P} = 1$ the core velocities are relatively small and that for $\mathbf{P} \to \infty$ they could be either small or large. The recent experiments by Siegwarth with ethylene glycol showing a relatively large variation of core temperature in the vertical direction seem to rule out a solution for $\mathbf{P} \to \infty$ with large core velocities. Therefore the equations might have been developed on the more restrictive condition of small core velocities. This reliance on experimental observation is not completely satisfactory, and it is desirable to develop theoretical arguments for assuming a particular flow condition in the core.

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Résumé—Les variations de la masse volumique d'un fluide s'écoulant dans un tube horizontal chauffé peut provoquer un écoulement secondaire ainso que des variations dans le gradient axial de pression à travers la section droite du tube. L'effet de l'écoulement secondaire sur le champ de température et sur l'écoulement primaire à la sortie d'un long tube chauffé électriquement ayant des parois épaisses de conductivité élevée est analysé dans le cas d'un nombre de Grashof–Prandtl élevé, pour lequel il existe près de la paroi une couche limite thermique mince. On expose un modèle pour le champ d'écoulement qui est cohérent avec l'observation expérimentale que les isothermes sont horizontales sur la plus grande longueur du tube. On trouve par un raisonnement dimensionnel que l'écoulement secondaire contrôle le flux de chaleur. Pour $P = 1$, l'écoulement primaire montre aussi un comportement de couche limite tandis que, pour $P \to \infty$, l'écoulement primaire est indépendant de l'écoulement secondaire. Pour une viscosité constante et un nombre de Prandtl infini, le nombre de Nusselt est directement proportionnel à la racine quatrième du produit de Grashof pour le nombre de Prandtl.

$$N = C_1 (GP)^{\frac{1}{4}}$$

On a calculé par les méthodes intégrales que $C_1 = 0{,}471$. Un bon accord est obtenu entre les calculs basés sur le modèle proposé et l'expérience.

DER EINFLUSS VON SEKUNDÄRSTRÖMUNGEN AUF DAS TEMPERATURFELD UND DIE PRIMÄRSTRÖMUNG IN EINEM BEHEIZTEN HORIZONTALEN ROHR.

Zusammenfassung—Dichteänderungen in einem Fluid, das durch ein beheiztes horizontales Rohr strömt, können sowohl Sekundärströmungen als auch Änderungen des axialen Druckgradienten über den Rohrquerschnitt verursachen. Der Einfluss der Sekundärströmung auf Temperaturfeld und Primärströmung am Auslass eines langen elektrisch beheizten Rohres mit dicken Wänden hoher Leitfähigkeit wird untersucht für dem Fall grosser Grashof–Prandtl Zahlen, bei denen eine dünne Temperaturgrenzschicht an der Wand existiert. Es wird ein Modell für das Strömungsfeld angenommen, das konsistent ist mit der experimentellen Beobachtung, dass nämlich über den grössten Teil des Rohres die Isothermen horizontal verlaufen. Aus Dimensionsbetrachtungen ergibt sich, dass die Sekundärströmung verantwortlich ist für die Grösse des Wärmestroms. Für $\mathbf{P} = 1$ zeigt die Primärströmung Grenzschichtverhalten, während für $\mathbf{P} \to \infty$ die Primärströmung von der Sekundärströmung abhängt. Für konstante Viskosität und sehr grosse Prandrlzahlen ist die Nusseltzahl direkt proportional der vierten Wurzel des Produkts aus Grashof- und Prandtlzahl.

$$\mathbf{N} = C_1(\mathbf{G} \, . \, \mathbf{P})^{\frac{1}{4}}$$

Aus integralen Lösungsmethoden ergibt sich die Konstante zu $C_1 = 0{,}471$. Die Berechnungen, denen das angenommene Modell zugrunde liegt, stimmen sehr gut mit dem Experiment überein.

ВЛИЯНИЕ ВТОРИЧНОГО ТЕЧЕНИЯ НА ТЕМПЕРАТУРНОЕ ПОЛЕ И ОСНОВНОЙ ПОТОК В НАГРЕТОЙ ГОРИЗОНТАЛЬНОЙ ТРУБЕ

Аннотация—Изменения плотности жидкости, текущей в нагреваемой горизонтальной трубе, могут вызвать вторичное течение, а также приводят к изменениям осевого градиента давления по сечению трубы. Анализируется влияние вторичного течения на температурное поле и основной поток на выходе из длинной электрически обогреваемой трубы с толстыми стенками высокой проводимости для большого числа Прандтля–Грасгофа, когда вблизи стенки существует тонкий температурный пограничный слой. Разработана модель поля потока, соответствующая экспериментально обнаруженному факту горизонтальности изотерм на большей части трубы. Методом размерностей найдено, что вторичный поток регулирует интенсивность теплообмена. При $\mathbf{P} =$ основной поток зависит от развития пограничного слоя, в то время как при $\mathbf{P} \to \infty$ он не зависит от вторичного течения. Для постоянной вязкости и бесконечного числа Прандтля число Нуссельта прямо пропорционально корню четвертой степени из произведения чисел Грасгофа и Прандтля

$$\mathbf{N} = C_1(\mathbf{G} \, . \, \mathbf{P})^{\frac{1}{4}}$$

Интегральными методами оценено значение $C = 0{,}471$. Между основанными на предложенной модели расчетами и экспериментами получено хорошее соответствие.